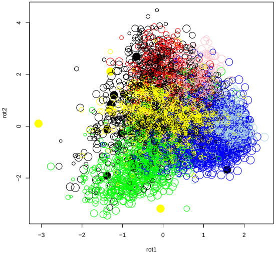


# Linear methods for large data

Dean Foster

Amazon

# This is a talk about some other people's paper

**"Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions"**

- by Halko, Martinsson, and Tropp.
- It is my current favorite paper.
- Today, I'll be applying it to a several problems in ML / statistics

# Basic method

problem Find a low rank approximation to a $n \times m$ matrix $M$.

solution Find a $n \times k$ matrix $A$ such that $M \approx AA^\top M$

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Construction $A$ is constructed by:

1. create a random $m \times k$ matrix $\Omega$ (iid normals)
2. compute $M\Omega$
3. Compute thin SVD of result: $UDV^\top = M\Omega$
4. $A = U$

# FAST MATRIX REGRESSIONS

# Using random methods for regression

Toy problem: $p \ll n$:

# Using random methods for regression

Toy problem: $p \ll n$:

- Solving least squares: (a la Mahoney)
  - Generates provably accurate results.
  - Instead of $np^2$ time, it runs in $np$ time.
  - This is fast! (I.e. as fast as reading the data.)

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  - Generates provably accurate results.
  - Instead of $np^2$ time, it runs in $np$ time.
  - This is fast! (I.e. as fast as reading the data.)
- But we should be unimpressed.
- Alternative fast (but stupid) method:
  - Do least squares on a sub-sample of size $n/p$
  - Runs in time $np$.
  - Same accuracy as the fast methods.

# A better fast regression

- Create "sub-sample" $\hat{X} \equiv AA^\top X$
- Estimate
$$\hat{\beta} = (\hat{X}^\top \hat{X})^{-1} X^\top Y$$
- (Mahoney also subsampled $Y$ and hence lost accuracy.)

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- As accurate as using all the data (NIPS 2013)

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# What about $p \gg n$?

- Sub-sample the other side of the $X$ matrix
- Generates a PCAs regression
- Sub-sample columns almost works
- Fast matrix multiply fixes the “almost” (NIPS 2013)
- Aside: yields fast ridge regression also (JMLR 2013)

# What about $p \approx n$?

- If $p$ is almost as large as $n$, then subsampling doesn't provide any speed up
  - To sub-sample enough to be accurate leaves the problem large
  - So best we could do would end up with a regularized answer

# What about $p \approx n$?

- But, if $p < n$, we might want not want to do regularized regression
- Yichao Lu and I combined these random matrix methods with gradient decent to find a good solution. (UAI 2014)

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- Yichao Lu and I combined these random matrix methods with gradient decent to find a good solution. (UAI 2014)
- Aside: Isn't gradient decent is a boring numerical analysis concept from Newton's age?
  - It has lately become the coolest thing in ML
  - For example, in Dec 2017, it solved 4 amazingly hard problems
  - No I won't tell you what they were since that would destract me from the rest of the talk:
    - Faces
    - Go
    - Automatic programming (aka tuning NMT)
    - Speech synthesis

Outline:

1. Least squares regression (we just finished).
2. Sparse Linear Regression (up next).
3. Fast CCAs.
4. Fast HMMs.
5. Fast parsing.
6. Fast clustering.

   ⋮

   (Zeno will kick in somewhere in this series)

All are connected to the fast matrix decomposition.

# (2) Sparse linear regression

# VIF regression

- Most variables are useless
- Why not only regress on the variables which provide value?
- Called stepwise regression
  - Often given fancier names (i.e. model selection)
  - Buzz words: AIC, BIC, RIC, IC, you get the pattern
- I'll discuss an algorithm called streaming feature selection

# VIF regression

- Basic method: Stream over the features, trying them in order
- Even more gready than stepwise regression (2006)
- Instead of orthogonalizing each new $X$, only approximately orthogonalize it. (2011)
  - Can be done via sampling
  - Can be done use fast matrix methods

# VIF regression

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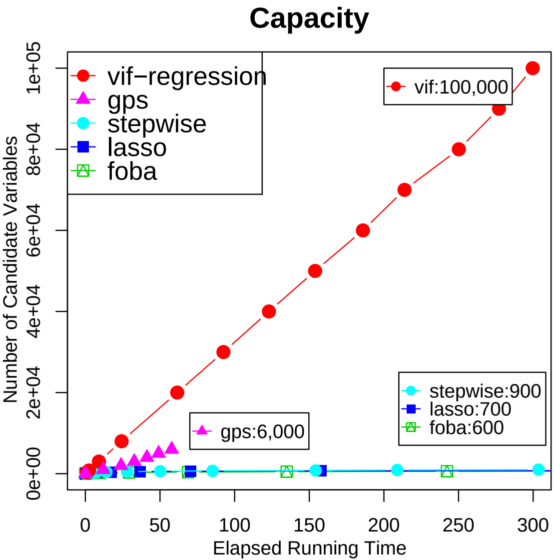
  - Can be done via sampling
  - Can be done use fast matrix methods
- Nice statistical properties:
  - For sub-modular problems, this will generate almost as good an estimator as best subsets. (2013)
  - provides mFDR protection (2008)

# VIF speed comparison



# (3) CCA for Semi-supervised data

# CCA: Usual data table for data mining

$$\begin{bmatrix} Y \\ \\ (n \times 1) \\ \\ \end{bmatrix} \begin{bmatrix} X \\ \\ (n \times p) \\ \\ \end{bmatrix}$$

with $p \gg n$

# With unlabeled data

$m$ rows of unlabeled data:

$$\begin{bmatrix} Y \\ \\ n \times 1 \\ \\ \end{bmatrix} \quad \begin{bmatrix} X \\ \\ (n+m) \times p \\ \\ \\ \\ \\ \end{bmatrix}$$

# With alternative X's

$m$ rows of unlabeled data, and two sets of equally useful $X$'s:

$$\begin{bmatrix} Y \\ n \times 1 \end{bmatrix} \quad \begin{bmatrix} X \\ (n+m) \times p \\ \\ \\ \end{bmatrix} \quad \begin{bmatrix} Z \\ (n+m) \times p \\ \\ \\ \end{bmatrix}$$

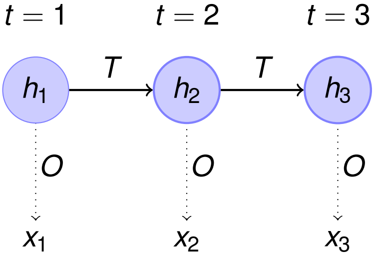
With: $m \gg n$

# Examples

- Named entity recognition
  - Y = person / place
  - X = spelling of the name itself
  - Z = words before target
- Modeling words in a sentence
  - Y = Current word
  - X = previous words
  - Z = future words
- Sitcom speaker identification:
  - Y = which character is speaking
  - X = video
  - Z = sound
- We will call these the multi-view setup
- Theory: Using the top few CCA directions is almost as good as the best linear model. (2006-14)

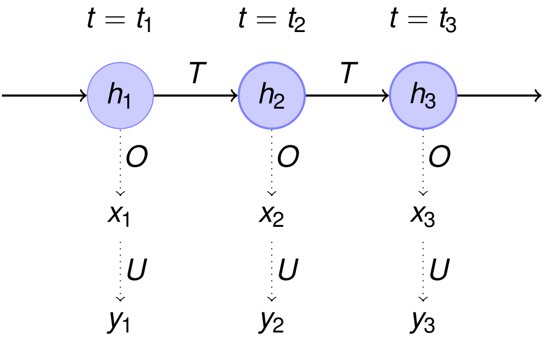
# (4) HMMs

# Hidden Markov Model



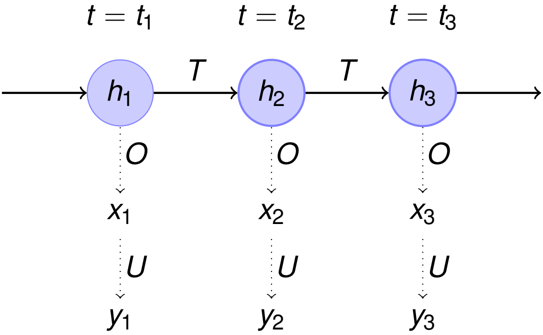

HMM with states $h_1$, $h_2$, and $h_3$ which generate observations $x_1$, $x_2$, and $x_3$.

# Hidden Markov Model




The $Y$'s are our eigenwords.

# Hidden Markov Model




The $Y$'s are our eigenwords.

$$\Pr(x_t, \ldots, x_1) = 1^T T \mathrm{diag}(OU^\top y_t) \cdots T \mathrm{diag}(OU^\top y_1) \pi$$

- Leads to "Eigenwords" a competitor of Google's more popular word2vec (2006-14)

# etc

- (5) Extends to trees:

# etc

- (5) Extends to trees:
  - We can extend the HMM material to dependency parsing
  - Same sample complexity (2012)
  - Raw MST Parser is 91.8% accurate
  - Adding eigenwords: 2.6% error reduction
  - eigenwords plus Re-ranking: 7.3% error reduction
  - Extended to constituent parsing (2014)

# etc

- (5) Extends to trees:
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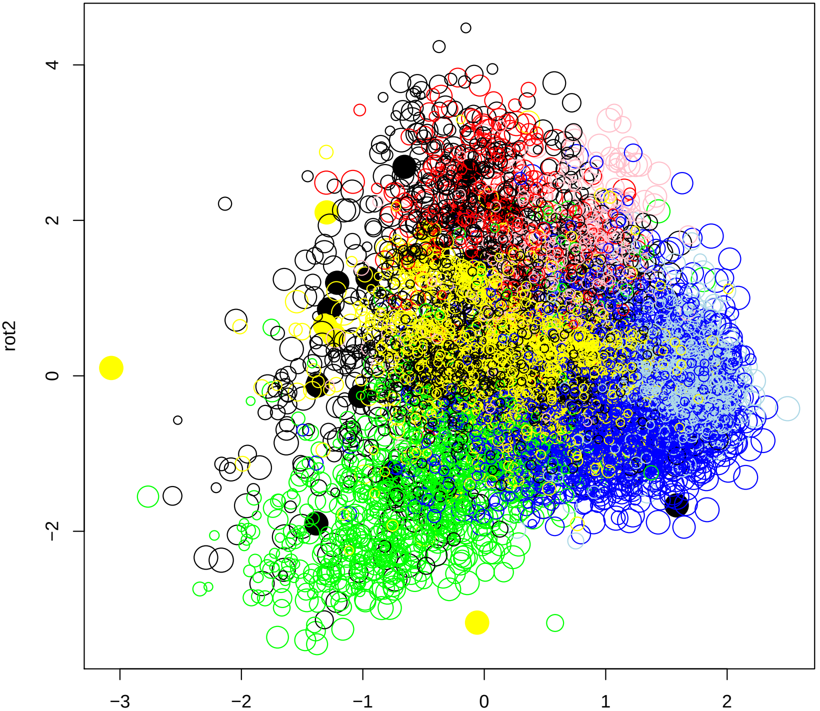
  - Extended to constituent parsing (2014)
- (6) Clustering:

**Theorem (with Hsu, Kakade, Liu, Anandkumar, NIPS 2012)**

*Maximizing $E(\mu^\top X)^4$ will find the natural coordinate system for LDA.*



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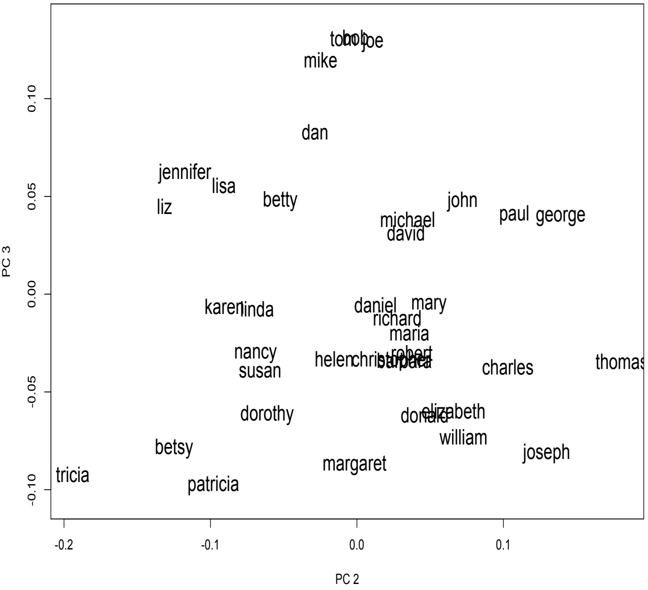

tom
bob
joe
mike
dan
jennifer
lisa
betty
liz
john
paul
george
michael
david
karen
linda
daniel
mary
richard
maria
nancy
helen
robert
barbara
charles
thomas
susan
dorothy
elizabeth
donald
william
joseph
betsy
margaret
tricia
patricia
0.10
0.05
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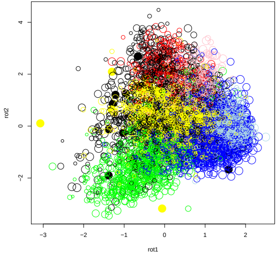
-0.05
-0.10
-0.2
-0.1
0.0
0.1
PC 2
PC 3

# Conclusions

- These new fast matrix methods are easy to program.
- They generate statistically useful results.
- So, read Halko, Martinsson, and Tropp!



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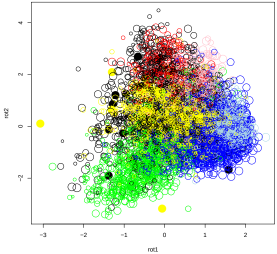


Thanks!

(For slides: google “dean foster” and search “linear.”)

# COAUTHORS

# Coauthors (Last updated 2015)

| | | U Penn | Elsewhere |
|---|---|---|---|
| Stat | faculty | Edward George<br>Robert Stine | Tong Zhang (Rutgers)<br>Sham Kakade (UW) |
| | students | Dongyu Lin (ATT)<br>Jordan Rodu (CMU)<br>Kory Johnson<br>Yichao Lu (Amazon) | |
| CS | faculty | Lyle Ungar | Michael Collins (Google)<br>Daniel Hsu (Columbia) |
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’15

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’11 ’12 ’13

’94 ’00 ’04 ’06 ’08 ’10 ’11 ’12 ’13

# Fast Regressions

**Theorem (with Yichao Lu, Parmaveer Dhillion, Lyle Ungar)**

*If $n > p^3$, then the algorithm defined by:*

- *Let $m = \sqrt{n}$*
- *Pull out a sub-sample of size m from $X$'s and call it $Z$.*
- *Let $\hat{\beta} \equiv (Z^\top Z)^{-1} X^\top Y$*

*then the CPU time is $O(np)$ and accuracy is as good as the usually estimator.*

# Fast Principal components regressions

**Theorem (with Yichao Lu, Parmaveer Dhillion, Lyle Ungar)**

*If $p > n$, then using a SRHT on the columns followed by regression will take $O(np\log(p))$ time and lose a constant factor on the statistical accuracy.*

# PCR is close to ridge regression

**Theorem (with Sham Kakade, Parmaveer Dhillion, Lyle Ungar)**

*A ridge regression can be quickly approximated by regressing on the top principal components. In particular, for a ridge parameter $\lambda$ using components with singular values larger than $\lambda$ will be within a factor of 4 of the ridge estimator on statistical accuracy. (*JMLR *2013)*

# Fast algorithm when $p$ is large

Problem: Regress $Y$ $X$, where $X$ has $p \approx n$. Hence we can't simply subsample the $X$ since it will still be too large.

1. Find a low rank (of dim $k$) approximation to $X$, call this matrix $P$.
2. Fit $Y$ to this approximation and compute the residuals $R$
3. Run a gradient least squares regression of $R$ on both $P$ and $X$.

This method can be thought of as a preconditioned gradient method. Yichao Lu and I (2015) showed that this has good performance for a wide variety of singular values for $X$.

# mFDR for streaming feature selection

Streaming feature selection was introduced in *JMLR* 2006 (with Zhou, Stine and Ungar).
Let $W(j)$ be the "alpha wealth" at time $j$. Then for a series of p-values $p_j$, we can define:

$$W(j) - W(j-1) = \begin{cases} \omega & \text{if } p_j \le \alpha_j \ , \\ -\alpha_j/(1-\alpha_j) & \text{if } p_j > \alpha_j \ . \end{cases} \tag{1}$$

**Theorem**

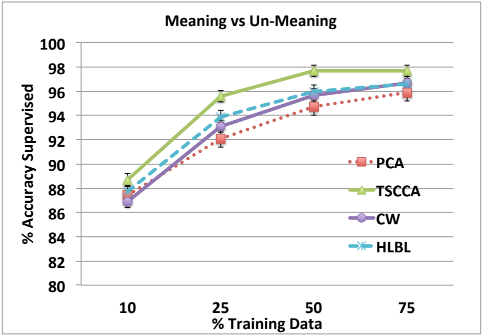
*(Foster and Stine, 2008,* JRSS-B*) An alpha-investing rule governed by (1) with initial alpha-wealth $W(0) \le \alpha\,\eta$ and pay-out $\omega \le \alpha$ controls mFDR$_\eta$ at level $\alpha$.*

# VIF regression

**Theorem**

*(Foster, Dongyu Lin, 2011) VIF regression approximates a streaming feature selection method with speed* $O(np)$.

# Eigenwords to estimate PERMA




See paper for the predictions of the other 4:

- *Positive emotion* (aglow, awesome, bliss, . . . ),
- *Engagement* (absorbed, attentive, busy, . . . ),
- *Relationships* (admiring, agreeable, . . . ),
- *Meaning* (aspire, belong, . . . )
- *Achievement* (accomplish, achieve, attain, . . . ).

(P. Dhillon, J. Rodu, D. Foster and L. Ungar., *ICML 2012*)

# Using a CCA between $X$ and $Z$ to generate features

We can compute a CCA between $X$ and $Z$ to find a good subspace to use to predict $Y$.

- CCA = canonical correlation analysis
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- CCA = canonical correlation analysis
- Finds directions that are most highly correlated
- Can be solved by doing successive regressions
- So, we can use our fast regression algorithms (2014)

Results:

- Theory: Using the top few CCA directions is almost as good as the best linear model. (2006)
- We can use this to generate Eigenwords (ICML 2012)

# Fast CCA method

(This is work in progress.) Yichao Lu has two current papers on this. The first shows how to use fast PCA and gradient decent to do a fast regression. The second shows how to use this successively to do a fast CCA. Kakade, Hsu and Zhang also have a fast CCA method, but it suffers from getting a less accurate answer than statistically optimal.

# Submodular

**Theorem**

*(Foster, Johnson, Stine, 2013) If the R-squared in a regression is submodular (aka subadditive) then a streaming feature selection algorithm will find an estimator whose out risk is within a factor of $e/(e-1)$ of the optimal risk.*

# HMM Results

- Sample complexity (2010)
- Empirical results in NLP
  - Named Entity Recognition (CoNLL ’03 shared task)
  - Chunking (CoNLL ’00 shared task)
  - Eigenwords added signal to state of the art systems for both tasks
  - (2011)
- Neural data (2013)

# HMM theorem

This is the first theorem we did for HMMs. We now have many other versions for parsing and extensions to continuous data.

**Theorem (with Rodu, Ungar)**

*Let $X_t$ be generated by an $m \geq 2$ state HMM. Suppose we are given a $U$ which has the property that range$(O) \subset$ range$(U)$ and $|U_{ij}| \leq 1$. Using $N$ independent triples, we have*

$$N \geq \frac{128 m^2 (2t+3)^2}{\epsilon^2 \, \Lambda^2 \sigma_m^4} \log\left(\frac{2m}{\delta}\right) \cdot \overbrace{\frac{\epsilon^2/(2t+3)^2}{(\sqrt[2t+3]{1+\epsilon}-1)^2}}^{\approx 1}$$

*implies that*

$$1 - \epsilon \leq \left| \frac{\widehat{\Pr}(x_1, \ldots, x_t)}{\Pr(x_1, \ldots, x_t)} \right| \leq 1 + \epsilon$$

*holds with probability at least $1 - \delta$.*

# CCAs

**Theorem**

*Let $\hat{\beta}$ be the Ridge regression estimator with weights induced by the CCA. Then under the multi-view assumption*

$$Risk(\hat{\beta}) \leq \left(5\alpha + \frac{\sum \lambda_i^2}{n}\right)\sigma^2$$

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$$Risk(\hat{\beta}) \leq \left( 5\alpha + \frac{\sum \lambda_i^2}{n} \right) \sigma^2$$

Estimator is least squares plus a penalty of:

$$\sum_i \frac{1 - \lambda_i}{\lambda_i} \beta_i^2$$

Where $\lambda_i$'s are the correlations

# CCAs

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*Let $\hat{\beta}$ be the Ridge regression estimator with weights induced by the CCA. Then under the multi-view assumption*

$$Risk(\hat{\beta}) \leq \left(5\alpha + \frac{\sum \lambda_i^2}{n}\right)\sigma^2$$

Multiview property $\alpha$ is the multiview property:

$$\begin{aligned} \sigma_x^2 &\leq \sigma_{x,z}^2(1+\alpha) \\ \sigma_z^2 &\leq \sigma_{x,z}^2(1+\alpha) \end{aligned}$$

- $5\alpha$ is the bias
- $\frac{\sum \lambda_i^2}{n}$ is variance

# Results on ConLL task

- Results on 2 NLP sequence labeling problems: NER (CoNLL '03 shared task) and Chunking (CoNLL '00 shared task).
- Trained on $\sim$ 65 million tokens of unlabeled text in a few hours!

Relative reduction in error over state-of-the-art:

| Embedding/Model | NER | Chunking |
|---|---|---|
| C&W | 15.0% | 18.8% |
| HLBL | 19.5% | 20.2% |
| Brown | 12.1% | 18.7% |
| Ando+Zhang | 5.6% | 14.6% |

"Multi-View Learning of Word Embeddings via CCA," *NIPS* 2011.

# Dependency parsing

In EMNLP 2012 (Rodu, Ungar, Dhillon, Collins) we extended the HMM results to dependency parsing.

# Review paper on parsing

We have a review paper: “Spectral Learning of Latent-Variable PCFGs,” with Cohen, Stratos, Collins, and Ungar, submitting to *JMLR*.

# Neural data: raw

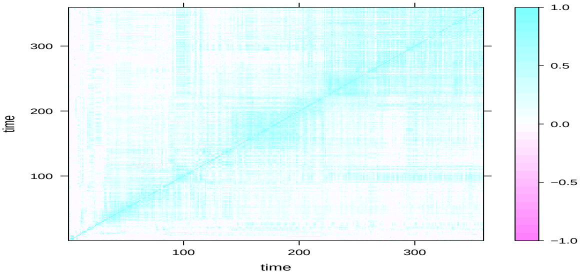


Figure 1: Correlation of raw observations, binned at 10 second bins

# Neural data: reduced dimension

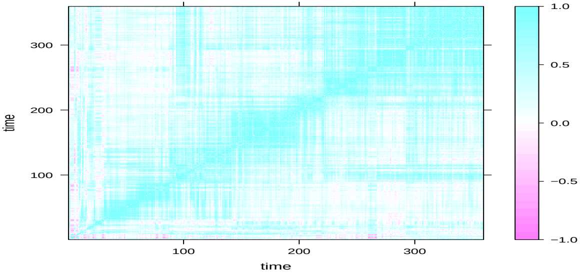


Figure 2: Correlations among reduced dimensional observations k=10

# Neural data: state estimate

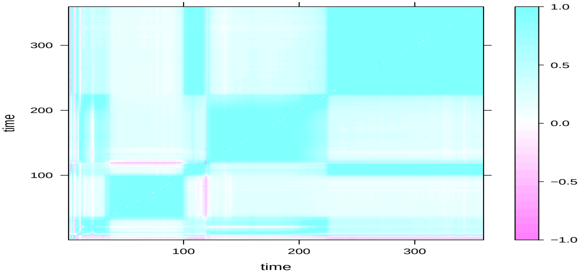


Figure 3: Correlations among the states of the system as time progresses k=10