Being Warren Buffett: A Classroom Simulation of Risk and Wealth when Investing in the Stock Market

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Abstract

Students who are new to Statistics and its role in modern Finance have a hard time making the connection between variance and risk. To link these, we developed a classroom simulation in which groups of students roll dice that simulate the success of three investments. The simulated investments behave quite differently: one remains almost constant, another drifts slowly upward, and the third climbs to extremes or plummets. As the simulation proceeds, some groups have great success with this last investment – they become the "Warren Buffetts" of the class, accumulating far greater wealth than their classmates. For most groups, however, this last investment leads to ruin because of its volatility, the variance in its returns. The marked difference in outcomes surprises students who discover how hard it is to separate luck from skill. The simulation also demonstrates how portfolios, weighted combinations of investments, reduce the variance. Students discover that a mixture of two poor investments emerges as a surprising performer. After this experience, our students immediately associate financial volatility with variance. This lesson also introduces students to the history of the stock market in the US. We calibrated the returns on two simulated investments to mimic returns on US Treasury Bills and stocks.

1. Introduction

The definition of variance as the expected squared deviation from the mean often strikes students as capricious. When shown a histogram, our students seldom suggest measuring spread by finding the average squared deviation from the mean – unless they have read the text or met the definition in other courses. Those who have seen boxplots pick the interquartile range and others suggest the average absolute deviation. Students often ask us "Why average squared deviations from the mean?" When teaching an introductory course, we cannot appeal to efficiency arguments that assume normality to justify variance as a "natural" measure of scale.

When dealing with money, however, the definition of variance is just right. In Finance, the risk of an investment is precisely the variance of its returns. Rather than link these through definitions, we have found it more engaging and memorable to let students experience the effects of variance first-hand in a simulation. In this experiment, students roll dice that determine the value of several investments and reveal the role of variance. The discussion of this simulation requires only basic properties of means and variances, with the most sophisticated property being that the variance of a sum of independent quantities is the sum of the variances.

We have used this dice simulation successfully for over 10 years in courses taught at three levels. Because the exercise requires relatively little background, it can be used early in the curriculum before normality and standard error. The simulation has become a standard component of the introductory undergraduate course in Statistics at Wharton. Students need only have been introduced to histograms and their connection to the mean, standard deviation, and variance. The idea of a discrete random variable is useful (for that is what the students will be simulating) but this is not necessary. We also regularly use this simulation in the required MBA course. MBA students generally have a better sense of the economics of investing, but many are nonetheless surprised to discover the rich connection between Statistics and Finance. In more advanced courses, such as undergraduate courses in mathematical statistics or probability, we use the dice simulation to illustrate discrete random variables. The simulation and ensuing discussion consume a full hour and 20-minute class; it also works well divided into 2 one-hour classes.

The investments simulated by the dice in this exercise mimic actual investments. One investment resembles a conservative money-market fund whose interest has been adjusted for the effects of inflation. At the other extreme, a second investment matches our intuitive definition of a risky stock. It resembles the performance of many of the high-flying tech stocks in the late 1990s during the dot-com bubble. A third lies between these extremes and performs like the overall stock market.

We have students simulate the value of these investments by rolling three differently colored dice. We label the three investments *Red*, *White,* and *Green* because it is easy to find dice in these colors. We have, on occasion, tried to save class time by using a computer to run the simulation; it is easy to program the simulation in Excel, say. End-of-the-term course evaluations have shown, however, that students who have done the experiment "by hand" more often mention this lesson as one that was particularly effective. Students not only see the importance of variance in Statistics, but they also discover the relevance of Statistics in the real world. After this simulation, everyone appreciates the importance of variance when looking at data. As our nation discusses privatizing Social Security and shifting retirement investments into stocks, it would be useful if more citizens understood these lessons.

The following section describes the dice simulation. The third section describes the origins of the simulated investments and explains how portfolios improve investments by reducing variation. This section also introduces the notion of volatility drag to quantify the effects of variation. The concluding section returns to the theme of distinguishing luck from skill.

2. The Dice Simulation

2.1 Getting Started

Before describing the simulation, we divide the class into teams. Teams of 3 or 4 students seem about right. One person on each team plays the role of nature (or the market) and rolls the dice. Another keeps track of the dice and reads off their values, and a third records the outcomes. Others can help out by retrieving the dice and checking the calculations.

Once we have divided the class into teams, we pose the following question. We've found it useful to elicit a written preference from each team before starting the simulation. This gets them talking about the simulation and avoids too many "Monday morning quarterbacks" in the subsequent discussion. If a team has chosen an investment before starting the simulation, the team members seem more interested in following their choice as the simulation evolves.

Question 1. Which of the three investments summarized in the following table seems the most attractive to the members of your group?

Investment	*Expected Annual Percentage Change*	*SD of Annual Percentage Change*
Green	8.3%	20%
Red	71%	132%
White	0.8%	4%

Table 1. Expected value and standard deviation of the annual percentage change in the value of three investments.

We describe the information in Table 1 using examples such as the following:

> Suppose that you invest $1000 in one of these choices, say *Red*. Table 1 tells you that you can expect the value of your investment to be 71% larger at the end of the first year, up to $1,710.

Similarly, if we start with $1000 in each of these, we'd expect to have $1,083 in *Green* and $1,008 in *White* after one year. Because the expected value of a product of independent random variables is the product of expectations, we can find the expectations for each investment over a longer horizon given this assumption. Over 20 years, the initial investment of $1000 in *Red* grows in expectation to an astonishing $\$1000 \times (1.71)^{20} = \$45,700,000$. By comparison, the initial investment in *Green* grows to $4,927 and *White* creeps up to $1,173.

Students find such calculations of expected values quite reasonable, but have little intuition for how to anticipate the importance of the standard deviation – other than to recognize that the presence of a large standard deviation means that the results are not guaranteed. The massive standard deviation leads some students to question the wisdom of investing in *Red*, but most find it difficult to see how to trade off its large average return for the variation. The annual return on *Red* is about 9 times that of *Green*, but its standard deviation is also 6.5 times larger. Few students appreciate the bumpy ride promised by *Red*.

At this point in the discussion, many teams find the large average return of *Red* quite appealing. Regardless of the level of the class, we have found the following example useful as a means of suggesting the impact of variation on the long-run behavior of an investment. Suppose that a graduate lands a good job that pays $100,000 per year. In the first year, the company does well and her salary grows by 10% to $110,000. The next year is leaner, and she has to take 10% cut in pay, reducing her salary down to $99,000. The average percentage change in her salary is zero, but the net effect is a loss of 1% of the starting salary over the two years. Figured at an annual rate, that's a loss of 0.5% per year. It turns out that this simple example is a special case of a more general property that captures how variance eventually wipes out investments in *Red.*

2.2 Running the Simulation

After this introduction, we pass out three dice to each team along with a data-collection form similar to that suggested in Figure 1. (A full-page version of this form suitable for use in class is available at www-stat.wharton.upenn.edu/~stine.) This form organizes the results of the simulation in a format useful in later steps. The unused last column saves space to compute the returns on a portfolio later in the exercise. We collect these sheets at the end of the simulation so that we can review the results in the next class. We have found that students keep better records when we tell them in advance that we will collect these forms at the end of class.

Round	**Green**	**Red**	**White**	
Starting value	$1000	$1000	$1000	
Return_1				
Value_1				
Return_2				
Value_2				
Return_3				

Figure 1. Initial rows of the data-collection form used to record the value of the three investments simulated by rolling a red die, a white die, and a green die.

Before the class begins the simulation, we carefully explain how the dice determine the values of the investments. Each roll of all three dice simulates a year in the market, and the outcomes of the dice determine what happens to the money held in each investment. Table 2 shows how the outcomes of the dice affect the values of the three investments. Each cell of Table 2 gives the *gross return* on every $1 invested. The gross return on an investment over some time interval is the ratio of its final value to its initial value. For example, if the green die rolls 1, then every dollar invested in *Green* falls in value to $0.80. Rather than use percentage changes as in Table 1, we switch to gross returns for doing the calculations in the simulation. Percentage changes are familiar, but returns are more natural for the simulation. We find it helpful to project this table onto a screen that is visible to the class as the simulation proceeds.

Outcome	**Green**	**Red**	**White**
1	0.8	0.05	0.95
2	0.9	0.2	1
3	1.1	1	1
4	1.1	3	1
5	1.2	3	1
6	1.4	3	1.1

Table 2. Annual gross returns for the three investments simulated by rolling three differently colored dice.

An example of the calculations explains the use of Table 2. Each investment in the simulation begins with an initial value of $1000, as indicated in the first row of the data collection form. To simulate a year in this market, each team rolls all three dice. Suppose that on the first roll the dice show these outcomes:

(Green 2) (Red 5) (White 3)

The value 2 for the green die tells us to use the gross return 0.9 from the second row of Table 2 for *Green*; *Green's* value becomes $1000 × 0.9 = $900. Said differently, the percentage change in *Green* is –10%. We have experimented by showing a table of percentage changes rather than returns, but we have found that students make fewer mistakes in calculations if given the returns. The values of the investments after the first year are:

Green $1000 × 0.9 = $900
Red $1000 × 3 = $3000
White $1000 × 1 = $1000

The outcomes of rolling all three dice a second time determine how these values grow or fall in the second "year" of the simulation. Assume that the second roll of the dice gives

(Green 4) (Red 2) (White 6)

In this case, the gross return for *Green* from the 4^{th} row of Table 2 is 1.1; *Green* increases by 10%. After two rounds, the three investments are worth

Green	$900 × 1.1	= $990
Red	$3000 × 0.2	= $600
White	$1000 × 1.1	= $1100

Figure 2 shows the recording sheet after the first two rolls.

Round	Green	Red	White	
Starting value	$1000	$1000	$1000	
$Return_1$	0.9	3	1	
$Value_1$	900	3000	1000	
$Return_2$	1.1	0.2	1.1	
$Value_2$	990	600	1100	
$Return_3$				

Figure 2. Data table recording the outcomes of the first two rounds of the dice simulation of three investments.

After illustrating how to use Table 2, we pose another question to the class and let the simulation begin.

***Question 2*. Which of the three investments described by the multipliers in Table 2 has the largest value after 20 years in your simulated market?**

We run the simulation for 20 "years" in order to allow the long-term patterns emerge. If the simulation runs much longer, the *Red* investment becomes less likely to do well. Stopping after 20 rounds leaves a good chance that some team will be doing very well with *Red*.

In a more advanced class that has covered discrete random variables, we describe Table 1 more precisely. Random variables are a natural way to represent the uncertainty of the value of investments that, unlike bank accounts, can increase or decrease in value. The random variable that is most natural in this context is the return on the investment. For example, let the random variable *R* denote the gross return on *Red* so that the value of *Red* after one year is \$1000 *R*. If the red die rolls 2, then Table 2 gives *R*=0.2 so that *Red* falls by 80%, dropping from \$1000 down to \$200. Similarly, the other columns in Table 2 define similar random variables, say *G* and *W*, for the returns on *Green* and *White*, respectively.

We assign an exercise to have more advanced students verify that the returns in Table 2 correspond to the means and standard deviations of the percentage changes shown in Table 1. For example, Table 1 implies that $E(R) = 1.71$, which can be verified from direct calculation:

$$\begin{aligned} E(R) &= \frac{0.05 + 0.2 + 1 + 3 + 3 + 3}{6} \\ &= 1.70833 \end{aligned}$$

The rest of the table follows similarly.

2.3 Pink

As the class runs the simulation, we browse the room to see how the different teams are doing and check that they are calculating the values correctly. Generally, the room gets a little noisy, particularly if there's a team for which *Red* is growing. *Red* triples in value half of the time, so there's a good chance that some team will do well with *Red* if the simulation is run 20 rounds. Direct calculation shows that the probability that *Red* ending with value \$10,000 or more is about 20%, and the chance for becoming a millionaire with *Red* is a bit larger than 5%.

After letting the class run the simulation for 20 rounds, we interrupt the chatter and pose another task. We ask the students to consider a hybrid investment that mixes the previous results for *Red* and *White*. We call this investment *Pink*. This task does not require more rolling of the dice. All of the information needed for this part of the experiment is already recorded on the data collection form.

To compute the value of *Pink*, we instruct the students to use the *previously recorded outcomes* of the red and white dice. It is again easiest to describe how to find the value of *Pink* with an example. *Pink* also begins the simulation with \$1000, evenly split between *Red* and *White*. As a result, the return for *Pink* is the average of the returns previously obtained for *Red* and *White*. For example, using the same outcomes as in the previous illustration (*Red*=5 for a return of 3 and *White*=3 for a return of 1), the value of the *Pink* becomes

$$\$1,000 \times \frac{3+1}{2} = \$2,000$$

Compounded in the second round (in which *Red*=2 with return 0.2 and *White*=6 with return 1.1), the value of *Pink* falls to

$$\$2{,}000 \times \frac{0.2 + 1.1}{2} = \$1{,}300$$

Figure 3 shows the data recording form after adding these calculations for *Pink* in the last column. It is essential that students average the *returns*, not the values, for *Red* and *White*.

Round	Green	Red	White	Pink
Starting value	\$1000	\$1000	\$1000	\$1000
$Return_1$	0.9	3	1	2
$Value_1$	900	3000	1000	2000
$Return_2$	1.1	0.2	1.1	0.65
$Value_2$	990	600	1100	1300
$Return_3$				

Figure 3. Data recording form with the values for Pink added to the calculations.

Before turning the teams loose to compute *Pink*, we pose a third group of questions to make them think before calculating. Because *Pink* mixes the returns of *Red* and *White*, most students anticipate it to mix bad with boring and fall in value.

***Question 3.* Before you compute your outcomes, discuss *Pink* with your team. How you expect *Pink* to turn out? Do you think it will be better or worse than the others?**

We have found it useful to circulate through the room as the teams figure out the results for *Pink*. Often, as they do the calculations, teams suspect that they have done something wrong because *Pink* does so well! A common mistake is to shortcut the work by averaging the final values for *Red* and *White*. This error gives a very different answer than obtained by averaging the returns.

2.4 Collecting the results

Once the teams have finished calculating the values of *Pink*, we query them in class for their results. To maintain flow of the discussion, we find it easiest to track the outcomes for the investments on a transparency that we augment as teams announce results. Figure 4 shows the results from a typical run of the dice simulation. *Pink* generally "wins" for most teams. *Green* shows steady growth and is usually the best of the original alternatives. *White* seldom moves far from the initial $1,000 stake. For most, *Red* bounces around and then becomes near worthless, falling to pennies in value.

Figure 4 about here

The dice simulation surprises most students, even more experienced MBAs. Perhaps the biggest surprise happens when a team announces a huge value for *Red*. As we poll the class, most teams announce small values in *Red*, often less than $1. It comes as quite a shock when a team announces that their investment in *Red* is worth more than $10,000,000. With a bit of fanfare, we proclaim this team to be the "Warren Buffetts" of the class. Business students generally know the name by reputation.

Warren Buffett is well-known among investors for his down-to-earth approach to investing. Often ignoring popular trends, Buffett built his company, Berkshire Hathaway, into a $133 billion holding company by purchasing the stocks of companies that made products he liked and understood. His strategy has been very successful. In 2005, *Forbes Magazine* estimated Buffett's net worth at $44 billion, second only to that of Bill Gates in its list of the most wealthy people in the world.

Pink presents the students with their second surprise. The results in Figure 4 are typical. *Pink* is more volatile than *Green*, but closes with a larger value. Across the class – with the exception of the Warren Buffetts – *Pink* usually finishes with the highest value. Though *Pink* mixes two investments that are individually poor choices, this simple mix of *Red* and *White* performs very well. That frequently seems impossible to the class: How can an average of two poor investments become so valuable? Their surprise brings curiosity and provides an incentive for trying to understand the role of the variance.

3. Discussing the Simulation

3.1 Why these multipliers?

We open our discussion of the dice game by linking the dice to real investments. *Green*, which does the best for most teams until they discover *Pink*, performs like the US stock market when adjusted for inflation. *White* represents the inflation-adjusted performance of US Treasury Bills, the canonical "risk-free" investment. We made up *Red*. We don't know of any investment that performs like *Red*. If you know of one, please tell us so we can make *Pink*!

We find our classes eager to see the underlying financial data. To save time, one could alternatively provide a handout summarizing these background facts. The timeplot in Figure 5 summarizes the history of stocks and Treasury Bills in the US, monthly from 1926 through the end of 2003. This plot tracks the cumulative value of one dollar invested in January 1926 in a value-weighted portfolio of the US stock market and 30-day Treasury Bills. (A value-weighted portfolio, such as the S&P 500, buys stock in proportion to their capitalized value. Alternatives such as the Dow-Jones Index simply buy one share of each.) The y-axis in the timeplot uses a log scale. When plotted on a log scale, geometric growth appears as a straight line.

Figure 5 about here

This plot is a little misleading because it ignores inflation. Although inflation has recently been low, it has exceeded 15% annually in the past. (For students who are unfamiliar with the impact of inflation, we find it helpful to show a timeplot of this series as well.) To adjust for inflation, Figure 6 shows the cumulative values after subtracting the rate of inflation from the growth of $1 investments in the stock market and Treasury Bills. To measure inflation, we used annual changes in the Consumer Price Index in the US. Once adjusted for inflation, investments in Treasury Bills declined for several long periods, and the initial $1 invested in Treasury Bills in 1926 closes at $1.66 at the end of 2003. The $1 invested in the stock market reaches $161, even allowing for the Great Depression and the bursting of the dot-com bubble.

Figure 6 about here

Returns are the key random variables in the dice simulation. Figure 7 shows monthly gross returns for stocks and Treasury Bills (after subtracting inflation) on the scale 0.7 to 1.4 (–30% to 140%). The variation of the return on Treasury Bills is much smaller than the variation in returns on stocks. Returns on the stock market use the full range of the plot; those of Treasury Bills never venture far from 1. We prefer to plot monthly returns rather than annual returns because monthly returns reveal features that are otherwise lost on the more coarse time scale. At the left of Figure 7, for example, is the Great Depression starting in the late 1920s and running through the 1930s. Returns on the market were incredibly volatile during that period. In 1933, the market dropped almost 30% in one month. Less well known is that about a year later, the market increased by about 40% in each of two months. Following WW II, gross returns on stocks became rather stable – at least in comparison to the volatility during the Depression. One can also see other important events, such as the fall of the stock market in October 1987.

Figure 7 about here

	Stocks	*T-Bills*
Mean	1.0877	1.0073
Std Dev	0.2050	0.0404
Variance	0.0420	0.0016
N	78	78

Table 3. Means, standard deviations and variances of annual, inflation-adjusted returns on US stocks and Treasury Bills.

Table 3 gives means, standard deviations, and variances of the annual historical returns on stocks and Treasury Bills. The annual return on the US stock market above inflation from 1926 through 2003 averaged 1.0877, slightly more than 8% with standard deviation 0.205, about 20%. Returns on Treasury Bills have been essentially flat, just keeping pace with inflation. The average return above inflation for Treasury Bills has been 1.0073, about 2/3 of one percent. The annual standard deviation is 5 times smaller than that of the stock market.

When compared to the properties of the investments simulated by the dice, we see that *Green* mimics the market and *White*, Treasury Bills. The expected return on *Green* (1.083, shown in Table 1 as 8.3%) is very close to the annual return on the market with comparable standard deviation. Similarly both the mean of *White* (1.008) and its standard deviation (0.04) are close to those of annual returns on Treasury Bills.

3.2 The Role of Variance: Volatility Drag

Pink surprises the class. Before getting into details, it is essential that students understand that *Pink* is not a simple average of *Red* and *White*. The returns on *Pink* average those on *Red* and *White*, but one does not get this performance by starting with $500 in *Red* and $500 in *White* and leaving it there. *Pink* requires that the portfolio be *rebalanced* at the end of each period so that half of the value is kept in *Red* and half in *White*. Returning to the illustrative calculations, at the end of the first round the initial $500 invested in *Red* grows to $1500 and the $500 invested in *White* holds its value. Before the next round, the portfolio needs to be put back into 50-50 balance; that is, we need to move $500 from *Red* into *White*, so that each has $1000 at the start of the next round. This "protects" some of the earnings produced by *Red* in the prior round from subsequent volatility. When the second roll wipes out

80% of the value of *Red*, it only reduces the $1000 left in *Red* down to $200. The other $500 produced by *Red* in the first round remains safely in *White*.

We next like to show that it is possible to anticipate the success of *Pink*. All we need are its mean and its variance. We can get these moments from those for *Red* and *White* given in Table 1 or, to more accuracy, Table 4. (We will describe the last column of Table 4 shortly.) Because the return on *Pink* averages those on *Red* and *White*, students willingly accept that the mean return on *Pink* is (1.7083 + 1.0083)/2 = 1.3583. Finding the variance is harder and requires that students know two rules for manipulating variances:

(a) Constants factor out with squares, $\text{Var}(c\,X) = c^2\,\text{Var}(X)$, and

(b) For independent random variables X and Y, variances of sums are sums of variances, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$.

Because we simulate the returns using separate dice, it should be clear that the returns on *Red* and *White* are independent. (Along with the invention of *Red*, independence of the returns in the dice simulation is a simplifying assumption that differs from the real world. Returns on real investments are usually correlated, complicating the analysis of a portfolio.) Using (a) and (b), the variance of returns on *Pink* are easily found to be

$$\begin{aligned} Var(Pink) &= Var\left(\frac{R+W}{2}\right) \\ &= Var\left(\frac{R}{2}\right) + Var\left(\frac{W}{2}\right) \\ &= \frac{Var(R) + Var(W)}{4} \\ &= \frac{1.7554 + 0.0020}{4} = 0.4393 \end{aligned}$$

Pink gives up ½ of the return on *Red* in return for reducing the variance by ¼.

Investment	Mean Return	Variance of Return	Volatility Adjusted Return
Green	1.0833	0.0381	1.0643
Red	1.7083	1.7554	0.8307
White	1.0083	0.0020	1.0073
Pink	1.3583	0.4393	1.1387

Table 4. Mean, variance and volatility adjusted return of the four simulated investments in the dice game.

At this point, the class still lacks a way of anticipating the success of *Pink* and the failure of *Red*. The answer lies in finding an expression that combines the mean with the variance to shows how variation eats away at the value of an investment. Because it reflects how variation (or volatility) diminishes the value of an investment, we refer to this adjusted return as the *volatility adjusted return*. This is also known in Finance as the long-run return on an investment. The formula for computing the volatility adjusted return is simple:

Volatility-adjusted return = Long-Run Return

= Expected Return – (Variance of Return)/2

The penalty for variation, ½ of the variance, is sometimes called the *volatility drag*.

The last column of Table 4 shows the volatility adjusted return for *Green, Red*, *White* and *Pink*. Not surprisingly, *Pink* is most attractive, with more than twice the volatility adjusted return of the stock market. On the other hand, the volatility adjusted return for *Red* is less than 1, showing that it ultimately loses value. Even though *Red* loses value for most teams, mixing it with *White* reduces the variance and produces a huge win. (As a little follow-up exercise, you might want to have students consider the following: What is the optimal mix of *Red* and *White*? That is, what proportions of *Red* and *White* produce the highest volatility-adjusted return?)

Depending on the level of the class, we spend more or less time describing the origins of the formula for the volatility adjusted return. After all, the units of the mean and variance do not match so it seems odd to subtract one from the other. In an introductory class that has covered discrete random variables, we can get this formula for the volatility drag from our simple example of variation in changes in salary. Think of the changes in salary as a random variable, with half of the probability on a return of 1.1 (up 10%) and the other half on 0.9 (down 10%). The expected value of this random variable is 1, and its variance is $0.1^2 = 0.01$. Now look back at the introductory example. The salary dropped by 0.5% per year, effectively returning 0.995, which is precisely 1 – ½(0.01). When corrected for volatility, each year of employment reduces the salary by half of the variance of the percentage changes even though the expected return is 1.

For an advanced undergraduate class in mathematical statistics or probability, we take this much further. Although we seldom use this material in an introductory class, we have found it useful to explain volatility drag to more interested students who are surprised to see the importance of various approximations that they have seen in better calculus courses. This explanation amounts to developing an approximation to the geometric mean using the first two moments of the underlying random variable, here the returns. (An accessible summary of this and related approximations, as well as their use in evaluating investments, appears in Young and Trent 1969). Label the initial value of an investment as W_0; W_0 = $1000 in the dice game. Label the return on an investment, say *Red*, during year t as R_t. The value at the end of the first year is $W_1 = W_0 R_1$. By the end of year T the value is

$$W_T = W_0 R_1 R_2 \cdots R_T$$

Taking logs converts this product to a sum that holds an average:

$$\begin{aligned} \log W_T &= \log W_0 + \sum_{t=1}^{T} \log R_t \\ &= \log W_0 + T \times \frac{\sum_{t=1}^{T} \log R_t}{T} \\ &\approx \log W_0 + T \times E \log R_t \end{aligned}$$

The last approximation only applies for large T by the weak law of large numbers. The expected value $E \log R_t$ is called the expected log return in Finance, yet another name for the long-run growth rate. To arrive at the volatility drag, we use the familiar approximation

$$\log(1+x) \approx x - \frac{x^2}{2}$$

and write $R_t = 1 + r_t$. (Some also call r_t the return, adding to the confusion between returns and percentage changes.) If we denote $E\, r_t = \mu_r$ and $\text{Var}\, R_t = \sigma^2_r$, then the approximation to the log allows us write

$$\begin{aligned} E \log R_t &= E \log(1 + r_t) \\ &\approx E\left(r_t - \frac{r_t^2}{2}\right) \\ &\approx \mu_r - \frac{\sigma_r^2}{2} \end{aligned}$$

Because r_t is small in practice, $\text{Var}(r_t) = E\, r_t^2 - (E\, r_t)^2 \approx E\, r_t^2$. Thus the value of this investment after T periods is approximately

$$
\begin{aligned}
W_T &\approx W_0\left(e^{\mu_r - \sigma_r^2/2}\right)^T \\
&\approx W_0\left(1+\mu_r - \sigma_r^2/2\right)^T \\
&\approx W_0\left(E\,R_t - \mathrm{Var}(R_t)/2\right)^T
\end{aligned}
$$

It is useful to compare this expression to the expectation itself, namely $E\,W_T = W_0\,(E\,R_t)^T$. Alternatively, one can avoid these approximations by making the strong assumption that returns follow a lognormal distribution. That argument, however, requires an assumption about the distribution of the returns that is hard to verify in practice. For students who have studied economics and utility theory, it may also be useful to observe that maximizing the geometric mean is not universally optimal (Samuelson 1971). Finally, in a very advanced class, one can use this discussion to motivate the importance of the Shannon-Brieman-MacMillan theorem (for example, see Chapter 15, Cover and Thomas 1991). But we'll not do that here!

4. Conclusion

What about those Warren Buffetts?

We developed this simulation to show off the importance of the variance in assessing the long-term value of investments. *Pink* illustrates how one can gain positive long-run returns by designing a portfolio that sacrifices expected returns to reduce the variance.

Having used this simulation in undergraduate and MBA classes for several years at Wharton, we have come to appreciate the important message conveyed by the Warren Buffetts of the class. These are the few teams that, unlike most others, end the simulation with *Red* reaching into the millions. It comes as quite a surprise to the rest of the class to discover, as we collect the final values from the teams, that some of their classmates have had huge success with *Red*. The differences are not slight either. For a team whose $1000 in *Red* has shrunk to pennies, it seems impossible that another team's investment in *Red* is worth more than $10,000,000 at the end of the simulation. After all, they all used the same rules to find the value of *Red*. Indeed, we formerly ran the simulation longer, hoping that volatility would wipe out these lucky winners. We have, however, come to realize that these surprises allow us to present the students with an important question.

What makes them believe that the real Warren Buffett was not just lucky? After all, with millions of investors seeking profits from the stock market, could it be that Warren Buffett simply "got lucky." There is usually considerable resistance from fans of the "Oracle of Omaha", but even they have to concede how difficult it is to separate a knowledgeable strategy from a lucky strategy. In the dice simulation, all of the teams use the same "strategy" for *Red* and rolled the dice themselves; nothing is hidden in a mysterious random number generator. They all start with $1000 in *Red*, but only a lucky few end the simulation appearing a lot smarter than the others. In the dice game, they can all see that it was simply luck that produced the Warren Buffetts.

We are careful not to say that Warren Buffett became successful by sheer luck. After all, much of his reputation was earned by conservative investments in established companies, such as Coca-Cola, that others had overlooked. We simply point out the difficulty in separating skill from luck, a problem that bedevils investors in hedge funds and requires methods outside the scope of this paper that we plan to describe elsewhere.

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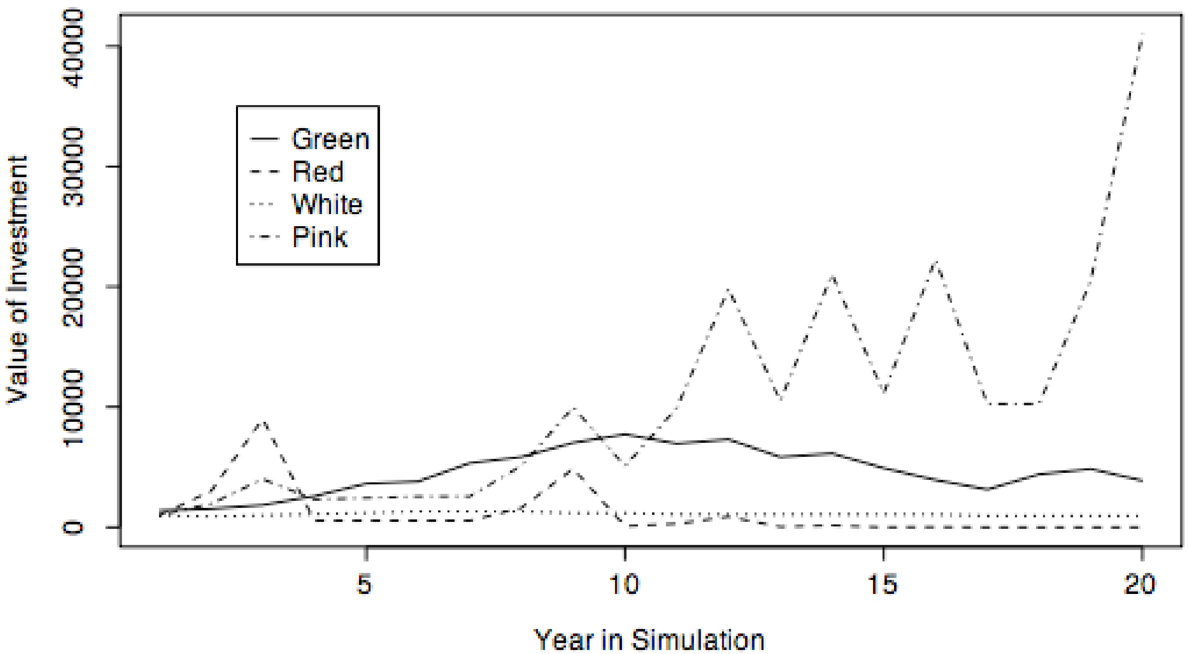

Figure 4. Timeplots of the values of four simulated investments in the dice game. The outcomes depict the typical results of the simulation.

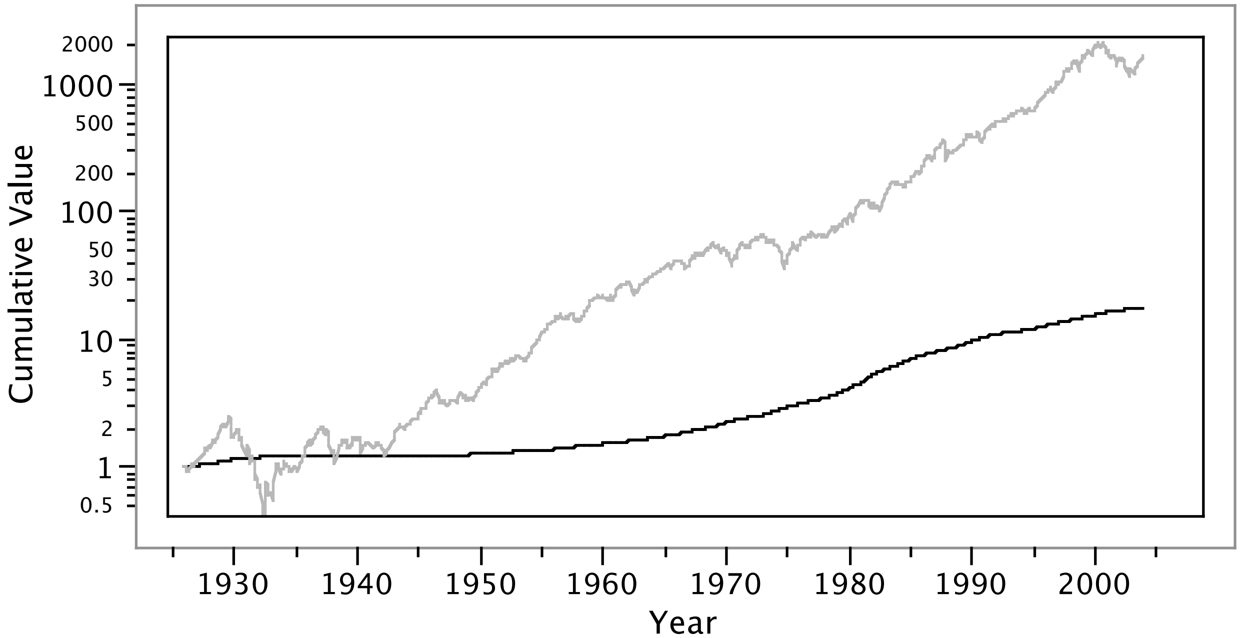

Figure 5. Cumulative value of investing $1 in 1926 in the stock market (gray) and in 30-day Treasury Bills.

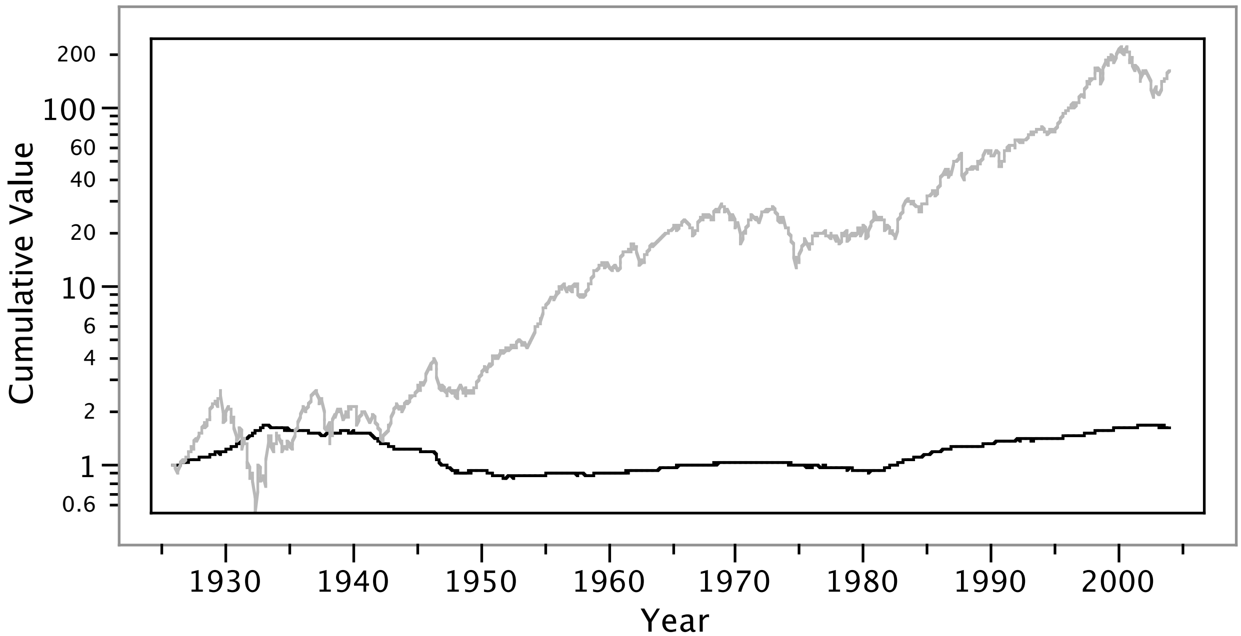

Figure 6. Cumulative value of investing $1 in 1926 in the stock market (gray) and in 30-day Treasury Bills after adjusting for inflation by subtracting the rate of change in the Consumer Price Index.

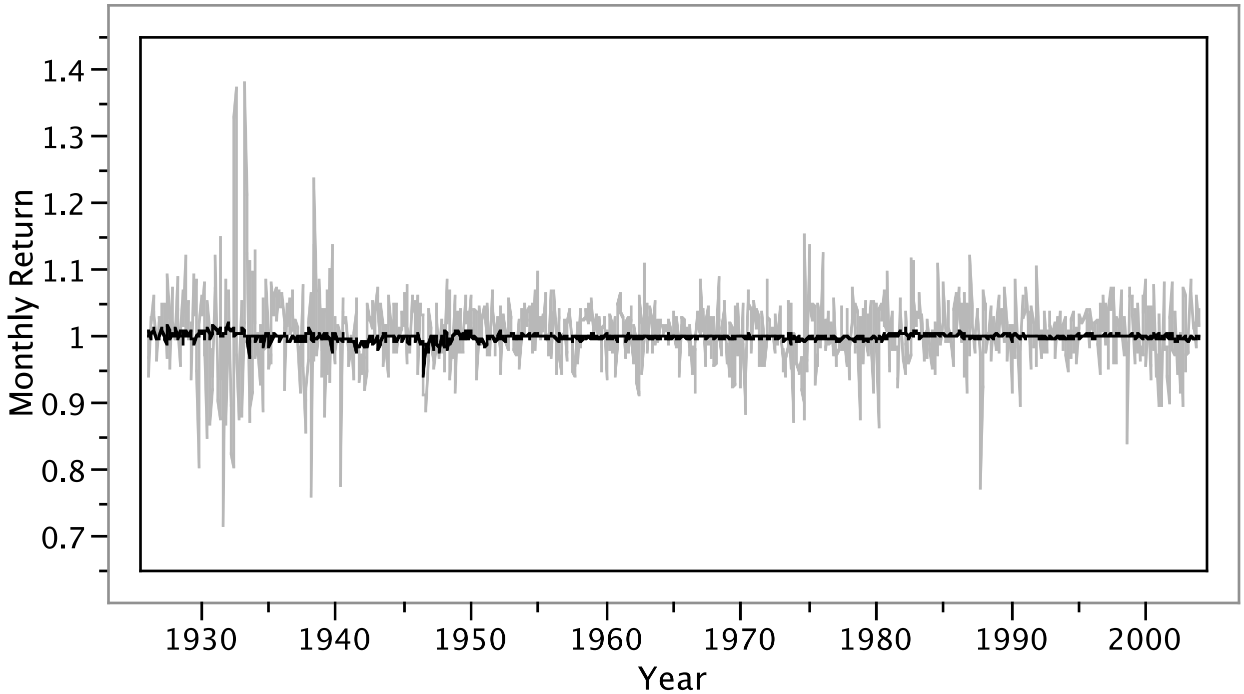

Figure 7. Timeplot of inflation-adjusted monthly returns for stocks (gray) and Treasury Bills.