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Operations Research, Volume 46, Issue 3 (May - Jun., 1998), 347-354.

Stable URL:
http://links.jstor.org/sici?sici=0030-364X%28199805%2F06%2946%3A3%3C347%3AAACOAC%3E2.0.CO%3B2-N

AN AXIOMATIC CHARACTERIZATION OF A CLASS OF LOCATIONS IN TREE NETWORKS

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(Received August 1991; revisions received December 1993, January 1995, November 1995, June 1996; accepted October 1996)

In this paper we describe four axioms that uniquely characterize the class of locations in tree networks that are obtained by minimizing an additively separable, nonnegative, nondecreasing, differentiable, and strictly convex function of distances. This result is analogous to results that have been obtained in the theory of bargaining, social choice, and fair resource allocation.

The single facility location problem on a network is the problem of selecting a point in a network so as to optimize an objective function that is distance dependent with respect to a set of points of the network. The problem has received a great deal of attention, as can be seen from the references in Hansen et al. (1987), Handler and Mirchandani (1979), and Mirchandani and Francis (1989). Most of the work in this area has been concerned with how to efficiently determine the point that optimizes the objective function. The conceptual issue of what is the appropriate objective function has received less attention. The choice of objective function tends to be ad-hoc and supported by attributes that are outside the ambit of the formal model. A number of researchers—Buhl (1988), Halpern and Maimon (1980), Holzmann (1990), McAllister (1976), and Morril and Symons (1977)—have investigated this question. This paper is also concerned with this issue. The approach taken will be the axiomatic one. In this we follow in the spirit of Buhl (1988), Holzmann (1990), and Vohra (1996).

So as to focus the discussion, we introduce some notation. Let T be a tree network with vertex set N and arc set A. Consider the arcs to be rectifiable, and for any two points x and y (whether in N or in the interior of an arc) in T we denote by $d(x, y)$ the length of the shortest path between them. Let (in an abuse of notation) $S = \{1, \ldots, n\}$ be a finite set of points in T. Associated with each point $i \in S$ is a nonnegative number w_i. Depending on the context, this number could represent the number of individuals at, or importance of, that point and is often called a weight. The general single facility location problem on a tree can now be stated as follows:

$$\min V(w_1 d(1, x), w_2 d(2, x), \ldots, w_n d(n, x))$$
$$\text{s.t.} \quad x \in T,$$

where V is a real valued function on $\mathbf{R}^n_+$.

If $V(w_1d(1, x), w_2d(2, x), \ldots, w_nd(n, x)) = \Sigma\{w_id(i, x): i \in S\}$, then, a point $x \in T$ that minimizes V is called an *absolute median* (Hakimi 1964). If, $V(w_1d(1, x), w_2d(2, x), \ldots, w_nd(n, x)) = \max\{w_id(i, x): i \in S\}$, then, a point $x \in T$ that minimizes V is called a *weighted absolute center* (Hakimi 1964). Both the absolute median and weighted absolute center have been extensively studied. Other choices of V that have been made are listed below along with the names of the points at which V attains a minimum.

(1) $\max\{d(i, x): i \in S\}$; *absolute center* (Hakimi 1964).
(2) $\mu\Sigma\{w_id(i, x): i \in S\} + (1 - \mu)\max\{d(i, x): \in S\}\}$ for $\mu \in (0, 1)$; *μ-cent-dian* (Halpern 1976).
(3) $\Sigma\{w_id(i, x)^t: i \in S\}$, $t > 1$; for $t = 2$ called a *squared median* (Vohra 1989b).
(4) $\Sigma_{i\in S}[w_id(i, x) - n^{-1}\Sigma\{w_id(i, x): i \in S\}]^2$; *variance point* (Halpern and Maimon 1980).

The list is not exhaustive. It is included to suggest the number and variety of choices that have been made for V. For a succinct review of the properties of and algorithms used to determine these points, see Hansen et al. (1987).

With so many choices, it may not be clear why one choice of V should be preferred over another. Clearly, the context in which the facility location problem is being solved will play some role. For example, suppose the facility we are siting provides a service whose quality is inversely related to the distance from the facility. Since our location is obtained by minimizing V, this would immediately rule out any V which is decreasing in $d(i, x)$ for all $i \in S$. These considerations alone are not sufficient to whittle the number of choices down to one. The axiomatic approach we take in this paper does not resolve this matter. It does, however, make explicit what is involved in choosing one function V over another.

Subject classifications: Facilities: location, discrete.
Area of review: Distribution.

0030-364X/98/4603-0347 $05.00

The axiomatic method starts by imposing axioms or principles that arg min $V(w_1d(1, x), w_2d(2, x), \ldots, w_nd(n, x))$ should satisfy. Notice we impose conditions on the point that will minimize V as opposed to V itself. We think it more natural to specify conditions on how a location behaves rather than on the function that will be used to measure the desirability of different locations. Second, a characterization of V alone, does not specify why one should minimize V rather than maximize. To decide one would have to appeal to some property that the resulting location should possess. Given the axioms, one deduces the class of, or unique V, that satisfies these axioms. Different axioms would characterize different objective functions. Thus, as Holzmann observes, "it becomes possible to argue in favor of, or against, a particular location in terms of principles satisfied or violated".

The axiomatic approach has been used successfully in a number of different areas, like social choice, Moulin (1988); bargaining, Lensberg and Thomson (1989); and cost allocation, Young (1985). In the context of facility location, the axiomatic approach was pioneered by Holzmann (1990), who identified axioms that uniquely characterize the squared median of a tree network. Subsequently, other axioms were proposed and used to characterize the absolute median, absolute center as well as squared median of a tree in Vohra (1996). Buhl (1988) has also argued the importance of the axiomatic approach in facility location, but the approach taken is different. He imposes axioms that characterize V rather than the location that is obtained by optimizing V.

Our goal in this paper is to describe a set of axioms that uniquely characterize the class of points in tree networks that are obtained by minimizing $V(w_1d(1, x), w_2d(2, x), \ldots, w_nd(n, x))$ when

$$V(w_1d(1, x), w_2d(2, x), \ldots, w_nd(n, x)) = \sum \{w_if(d(i, x)): i \in S\},$$

where f is a nonnegative, nondecreasing, differentiable, and strictly convex function. The squared median is an example with $f(x) = x^2$. This class of location functions is tree independent in the sense that it depends only on the distance between the points in S and not the "shape" of the specific tree which contains them. It excludes, for example, the following location rule:

> "If the underlying tree is a star, locate at the absolute center, otherwise locate at the squared median". Call this particular location function the **quasi-squared median**. We give a rigorous definition of this notion of tree independence later.

Before continuing we note that the restriction of our efforts to tree networks is not uncommon in the facility location literature. We have been unable to extend our proof technique to the case of more general networks. In the last section of this paper we discuss some of the difficulties associated with such a task. We note that results parallel to the kind we obtain here have been obtained in the context of bargaining, Lensberg (1987); apportionment of taxes, Young (1987); and social choice, Young (1976).

THE AXIOMS

Let S be a finite set of points in the tree T. Associated with each $x \in S$ is a nonnegative number, $w(x{:}S)$, say. Let $w(S) = \{w(x{:}S): x \in S\}$. We will call the pair $[S, w(S)]$ a *customer set*. By a *location rule in T* we shall mean a rule L that associates with each customer set $[S, w(S)]$ a unique point $L([S, w(S)])$ in T. If $x \in T$ is such that $x = L([S, w(S)])$ we will say that L *selects x with respect to* $[S, w(S)]$. In defining L we have restricted it to being single valued. Further notice that there is more than one set S which can be used to describe a given distribution of customer positions. These sets differ in having different points being assigned a weight of zero. For example, let S be the set consisting of a single point x with weight 1 and Q be the set consisting of two points x and y, with a weight of 1 at x and 0 at y. We will treat different descriptions of the same customer distribution as being equivalent. So, we would treat S and Q above as the same sets. We will assume that each customer prefers to have the location as close as possible to themselves. The first condition we impose on L is the following:

Unanimity (U). *If the customer set $[S, w(S)]$ consists of a single point x, then, $L([S, w(S)]) = x$.*

If $[A, w(A)]$ and $[B, w(B)]$ are two customer sets we denote the combined customer set by $[A \cup B, w(A \cup B)]$ where (in an abuse of notation) we define $w(A \cup B) = \{w(x{:}A) + w(x{:}B): x \in T\}$. The second condition we impose on L is

Consistency (CS). *If $[A, w(A)]$ and $[B, w(B)]$ are two customer sets with the property that $L([A, w(A)]) = L([B, w(B)])$, then $L([A \cup B, w(A \cup B)]) = L([A, w(A)])$.*

The consistency axiom captures the following notion. Imagine two separate groups of individuals (say Republicans and Democrats) $[A, w(A)]$ and $[B, w(B)]$. If both groups independently deem the point x as an acceptable location, then, x should be acceptable to the group formed by combining them. This axiom was first introduced by Young (1974) in the context of voting. The motivation in that context is the following. Imagine Congress and the Senate separately deciding in favor of bill A over bill B, but in joint session they reverse themselves. A voting rule that satisfied consistency would prevent such an occurrence. In this sense the main result of this paper can be seen as a "locational" analog of the results in Young (1974, 1976).

There are a wealth of location rules that satisfy U and CS, and so these two axioms are not sufficient to reduce the choices available to just one class. Further restrictions need to be placed on the choice of an acceptable location. The next axiom is a regularity condition that captures the notion that L should not be "overly sensitive" to changes in the weights, $w(S)$.

Continuity (CN). *For fixed S, $L([S, w(S)])$ is continuous in each $w(x{:}S) \geqslant 0, x \in S$.*

One of the simplest of nontrivial customer sets that must be dealt with is when $|S| = 2$. It seems reasonable to suppose that ones intuition about what is fair or reasonable in selecting a location should be clearest in this situation. The next two axioms place restrictions on L in the way it selects a point in this situation. For notational purposes we define a *simple customer set*, $[S, w(S)]$, to be one where S consists of just two points. Call the two points LEFT and RIGHT. If $[S, w(S)]$ is a simple customer set we will write it as $[w_1, w_2 \mid D]$ where:

(a) we will sometimes refer to the points LEFT and RIGHT as 1 and 2, respectively;
(b) w_j, $j = 1, 2$, is the weight at point j; and
(c) D is the distance between points 1 and 2, that is, $D = d(1, 2)$.

Tree Independence (TI). *Let T_A and T_B be any two trees containing simple customer sets $[A, w(A)]$ and $[B, w(B)]$ respectively. If $w_1(A) = w_1(B)$ and $w_2(A) = w_2(B)$ and the distance between 1 and 2 in both trees are the same, then the rule L selects the same point in the sense that*

$$d(1, L([A, w(A)])) = d(1, L([B, w(B)])), \text{ and}$$
$$d(2, L([A, w(A)])) = d(2, L([B, w(B)])).$$

The axiom TI excludes the quasi-squared median.

Lemma. *If TI is satisfied, $L([w_1, w_2 |D])$ will be some point between LEFT and RIGHT.*

Proof. Consider first a tree, T_A, that is a single path of length D. Let LEFT and RIGHT be the left and right endpoints, respectively, of this path. Assign a weight of w_1 to LEFT and w_2 to RIGHT. By definition, $x = L([w_1, w_2 |D])$ must be between LEFT and RIGHT.

Now consider any other tree T_B, and a simple customer set $[B, w(B)]$ such that $w_1(A) = w_1(B)$ and $w_2(A) = w_2(B)$ and the distance between 1 and 2 is D. By TI

$$d(1, L([B, w(B)])) = d(1, L([A, w(A)])) = d(1, x),$$

and

$$d(2, L([B, w(B)])) = d(2, L([A, w(A)])) = D - d(1, x).$$

These two equations imply that $L([B, w(B)])$ must be between LEFT and RIGHT even in T_B. □

Thus, we can represent $L([w_1, w_2 |D])$ by its distance from the point LEFT. From now on, L for a simple customer set will be specified by the distance from LEFT. Given TI, the axiom U is redundant. Simply take $w_2 = 0$, then pick an equivalent customer set S which does not include the point RIGHT and now use TI to reduce the tree to the singleton LEFT. Thus, the only possible point to locate at is the LEFT point (we include the axiom U, so as to make some of the proofs that follow clearer).

Population Monotonicity (PM). *Suppose $[S, w(S)]$ is a simple customer set and $x = L([S, w(S)])$, but not LEFT or RIGHT. Then $d(1, L([S, w(S)]))$ is strictly increasing in w_2.*

Informally, increasing w_2 shifts $L([S, w(S)])$ to the right. Notice that PM does not assume that $L([w_1, w_2 |D])$ is between LEFT and RIGHT. This would happen if L satisfied TI as well.

The main result of this paper is

Theorem. *A location rule L satisfies TI, CS, PM, and CN iff $\exists$ a nondecreasing, nonnegative, differentiable, strictly convex function f such that:*

$$L([S, w(S)]) = \arg\min \left[\sum \{w(y{:}S) f(d(x, y)) : y \in S\} | x \in T\right]$$

for all customer sets $[S, w(S)]$.

If we drop the restriction that L is single valued (and modify the statements of the axioms appropriately), then the theorem holds with strict convexity of f replaced by convexity. As the proof involves no new ideas, but is more tedious, we omit it. An example of a location rule that is not single valued is the absolute median.

INDEPENDENCE OF THE AXIOMS

In this section we establish that each of our axioms is independent of the others. This shows that each of TI, CS, CN, and PM is required to characterize the class of locations we are interested in. We exhibit, for each axiom, a location rule that satisfies all but that axiom.

It is a simple matter to construct a location rule, L, that satisfies TI, PM, and CN but not CS. An example is the variance point. A location rule that satisfies TI, CS, and PM but not CN is a little harder. To exhibit such a rule let $M([S, w(S)])$ be the set of points x that minimize $\Sigma\{w(y{:}S)d(y, x){:}y \in S\}$. Define L as follows:

$$L([S, w(S)]) = \arg\min \left[\max \{w(y{:}S)d(y, x){:}y \in S\} |x \in M([S, w(S)])\right].$$

It is straightforward to verify that this L satisfies PM, CS, and TI. To see that it does not satisfy CN consider the customer set $[w, w |D]$ for some $D > 0$. Now, $L([w, w \mid D]) = D/2$ but $L([w, w + s \mid D]) = D$ for any $s > 0$ no matter how small.

A location rule that satisfies CN, CS, and TI but not PM is

$$L([S, w(S)]) = \arg\min \left[\max \{d(i, x) : i \in S, w(i{:}S) > 0\} |x \in T\right].$$

Finally, a location rule that satisfies all the axioms except TI is the quasi-squared median.

The Proof

We begin by showing that any location rule F that can be represented by

$$\arg\min\,[\sum\{w(y{:}S)f(d(x,y)){:}\ y \in S\}|x \in T],$$

where f is a nondecreasing, nonnegative, strictly convex, differentiable function satisfies TI, CS, PM, and CN. Notice that F is well defined because we are minimizing a continuous function over a compact set. This constitutes the "easy" part of the theorem. To do this, we need a characterization of the points x in T that minimize $\Sigma\{w(y{:}S)f(d(x,y)){:}\ y \in S\}$. The following proposition allows us to do this.

If C is a connected component of T we will denote by $C \cap S$ those points of S in C. Let h be any nonnegative function from the reals. A point $x \in T$ will be called *h-stable* with respect to $[S, w(S)]$ iff

$$\sum\{w(y{:}S)h(d(x,y)){:}y \in C \cap S\}$$
$$\leqslant \sum\{w(y{:}S)h(d(x,y)){:}y \notin C \cap S\},$$

for all connected components of C of $T\backslash x$. In the case when T is a path, $T\backslash x$ consists of just two connected components. In that case h-stability implies:

$$\sum\{w(y{:}S)h(d(x,y)){:}y \in C \cap S\}$$
$$= \sum\{w(y{:}S)h(d(x,y)){:}y \notin C \cap S\},$$

where C is either one of the connected components.

Proposition 1. *Let f be a nonnegative, nondecreasing, differentiable, and strictly convex function and f' its derivative. Then,* $x = \arg\min\,[\Sigma\{w(z{:}S)f(d(z,y)){:}\ z \in S\} \mid y \in T]$ *iff x is f'-stable with respect to* $[S, w(S)]$.

Proof. We omit the proof (but provide some intuition) as it is a straightforward generalization of arguments to be found in Vohra (1989), for example. It is also a special case of a more general theorem in the theory of variational inequalities (see Kinderlehrer and Stampacchia 1980). The algorithmic implications are discussed in Hooker (1989).

To get some insight into why this proposition is correct (because it will be useful later) it is helpful to consider a special case. Let T consist of a straight line of length D with weight w_1 at one endpoint (LEFT) and weight w_2 at the other endpoint (RIGHT). Suppose we seek the squared median for this configuration:

$$\arg\min\,[w_1 d(1,x)^2 + w_2 d(2,x)^2{:}x \in T].$$

Rather than compute this quantity analytically, we design a system of springs that will produce the result. Take an ideal spring with Hooke constant w_1 (a number that measures the stiffness of the spring) and attach one of its ends to LEFT. Take another ideal spring with Hooke constant w_2 and attach one of its ends to RIGHT. Next, join the free ends of the two springs to each other and call the join p. Lay these springs down on the line T. Wait till they are in equilibrium, and look for where p is positioned on T. This will be the squared median.

Why is this the case? The equilibrium position is one that minimizes the total energy of the system. The amount of energy stored in a spring stretched to a length y is proportional to y^2. The constant of proportionality being half the Hooke constant of the spring. Hence, if p is at the point x in T, the energy stored in the system of springs is:

$$(1/2)[w_1 d(1,x)^2 + w_2 d(2,x)^2].$$

Clearly, at equilibrium, p will position at the squared median.

Notice also that in equilibrium the force of the spring pulling p to LEFT should be equal to the force pulling p to RIGHT. To stretch a spring a length of y requires a force proportional to y, the constant of proportionality being the Hooke constant of the spring. So, if p is at position x in T, the magnitude of the force pulling p to LEFT is $w_1 d(1, x)$. Similarly, the force pulling p to RIGHT is of magnitude $w_2 d(2, x)$. Hence, at equilibrium:

$$w_1 d(1,x) = w_2 d(2,x).$$

Notice that this last expression is simply the stability condition for the special case when T is a line and S is a simple customer set.

More generally, we think of the function we are trying to minimize as being the energy of a suitably configured system of springs and the stability condition as capturing the idea that at equilibrium the forces in one direction cancel out the forces in the other directions. □

The strict convexity of f implies that F is single valued (see Hansen et al. 1987 for example). That F satisfies CN follows from the fact that we minimize a continuous function over a compact set to obtain $F([S, w(S)])$. That F satisfies TI is clear. CS follows from the additive separability of F. We now show that F satisfies PM. Let $[w_1, w_2\,|D]$ be a simple customer set. By Proposition 1, the rule F selects a point x between points 1 and 2 such that:

$$w_1 f'[d(1,x)] = w_2 f'[d(2,x)].$$

If we denote the distance of x from point 1 by t, then:

$$w_1 f'(t) = w_2 f'(D-t).$$

Now increase w_2 to some $w > w_2$. Then, F must select a point at a distance y from 1, where y satisfies:

$$w_1 f'(y) = wf'(D-y).$$

If $y > t$, then PM is satisfied. Suppose not. Since f is strictly convex and nondecreasing, it follows that f' is increasing. Hence,

$$wf'(D-y) = w_1 f'(y) \leqslant w_1 f'(t) = w_2 f'(D-t)$$
$$< wf'(D-t) \Rightarrow D - y < D - t \Rightarrow y > t,$$

contradiction.

To prove the "hard" part of the theorem we begin with a sequence of propositions that establish its correctness in the case of simple customer sets of the form $[w_1, w_2\,|D]$.

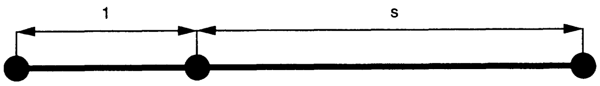

Figure 1. Configuration used to define $g(s)$.

Proposition 2. *$L([\mu w_1, \mu w_2 |D]) = L([w_1, w_2 |D]) \;\forall\; \mu \in R_+$ and D, w_1 and $w_2 > 0$.*

Proof. Suppose first that μ is a positive integer. Then, by producing μ copies of the customer set $[w_1, w_2 |D]$, combining them and invoking consistency we deduce that $L([\mu w_1, \mu w_2 |D]) = L([w_1, w_2 |D])$. Now suppose that μ is rational. Then $\exists$ two positive integers a and b such that $\mu = a/b$. Let $x = L([w_1/b, w_2/b |D])$. Taking a copies of $[w_1/b, w_2/b |D]$ we deduce that $L([aw_1/b, aw_2/b |D]) = x$ as well. Similarly, by taking b copies of $[w_1/b, w_2/b |D]$ we deduce that $x = L([w_1, w_2 |D])$. Hence,

$$L([\mu w_1, \mu w_2|D]) = L([w_1, w_2|D]).$$

Finally, suppose μ is irrational. Then $\exists$ a sequence of rational numbers, $\{\mu_n\}$, say, such that $\mu_n \to \mu$ as $n \to \infty$. By CN,

$$L([\mu_n w_1, \mu_n w_2|D]) \to L([\mu w_1, \mu w_2|D]),$$

however, $L([\mu_n w_1, \mu_n w_2 |D]) = L([w_1, w_2 |D]) \;\forall\; n$. This proves the proposition. □

Proposition 3. *$L([w, w |D]) = D/2 \;\forall\; w$ and $D > 0$.*

Proof. Suppose we put a weight of w_1 at LEFT and a weight of w_2 at RIGHT (a distance D from LEFT). Let L select a location at a distance t from LEFT. This location must be at a distance $D - t$ from RIGHT. Now switch the positions of the two weights. By TI, L must now select a location at a distance of $D - t$ from LEFT. Hence, $L([w_1, w_2 |D]) = D - L([w_2, w_1 |D]) \;\forall\; w_1, w_2 > 0$. The result follows with $w_1 = w_2 = w$. □

The main result of this paper is that a location rule L that satisfies CS, CN, PM and TI, can be represented by a convex function f. To obtain this function f we define a function g: $R \to R$ as follows:

$$g(s) = w \quad \text{iff } L([w, 1|1 + s]) = 1,\ s \geqslant 0.$$

Thus, $g(s)$ is the weight to be placed at the point LEFT so that if a weight of 1 is assigned to a point at a distance $1 + s$ from LEFT, L selects a location at a distance of one unit to the right of LEFT. This is illustrated in Figure 1.

So as to better understand the role of the function g, recall the spring analogy introduced in the proof of proposition 1. We have a location rule L and all we know is that it satisfies CS, CN, PM, and TI. We believe that it generates locations that coincide with those obtained by minimizing an additively separable convex function of distances. If this is the case we should be able to put together a collection of springs of appropriate "stiffness" so that the equilibrium position of the resulting configuration coincides with the location generated by L. To start with, we

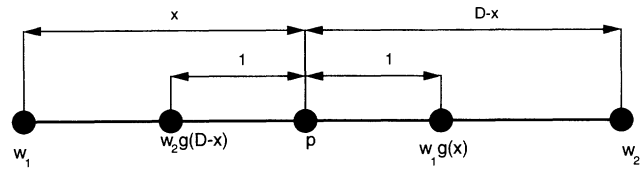

Figure 2. Configuration used to prove Proposition 4.

look at the simplest case of all: a two-customer set with $D = 1$ and $w_1 = w_2 = 1$. In this situation we know that L selects a location at a distance 1/2 from LEFT (recall Proposition 3). So, we need only two springs of equal stiffness to mimic this. Now suppose we move RIGHT a distance s away from LEFT (causing $D = 1 + s$) and simultaneously increase the weight at LEFT. If we are careful, we can do this and have L select a location at a distance of one unit from LEFT. However, in moving RIGHT we stretch the spring attached to LEFT so causing the equilibrium position to shift. In particular it may not be at a distance 1 unit from LEFT. To prevent this we need to increase the stiffness of the spring attached to LEFT. The function g represents the desired stiffness of the spring attached to LEFT.

In order to show that the function g is well defined we need to show that for every $s \geqslant 0 \;\exists$ a unique $w \geqslant 0$ such that $L([w, 1|1 + s)] = 1$. Let $u(w, s) = L([w, 1|1 + s])$. By Proposition 2, $u(w, s) = L([1, 1/w|1 + s])$. If we let w tend to infinity we have by U and CN that $L([1, 1/w|1 + s])$ tends to 0. Hence, for any $\epsilon > 0$ there is a w sufficiently large such that $u(w, s) < \epsilon < 1$. Now U implies that $u(0, s) = 1 + s \geqslant 1$ because the customer set with weight 0 at LEFT and weight 1 at RIGHT is the same as the customer set with weight 1 at RIGHT alone. Since $u(w, s)$ is continuous in w by CN, it follows by the intermediate value theorem that there is a w such that $u(w, s) = 1$. As $u(w, s)$ is strictly decreasing by PM, the w such that $u(w, s) = 1$ is unique.

The **integral of g is the convex function f we seek.** For example, if L were the squared median, it is easy to see from proposition 1 that $g(s) = s$. The integral of $g(s)$ would be $s^2/2$, the convex function we seek.

Proposition 4. *Consider the simple customer set $[w_1, w_2 |D]$. Then $L([w_1, w_2 |D]) = x$, if and only if $w_1 g(x) = w_2 g(D - x)$.*

Proof. Suppose first that $x = L([w_1, w_2 |D])$ but $w_1 g(x) \neq w_2 g(D - x)$. Without loss of generality we may assume that $w_1 g(x) < w_2 g(D - x)$. Let p be a point at distance x to the right of point 1 (see Figure 2). Place a weight of $w_1 g(x)$ one unit to the right of p and a weight of $w_2 g(D - x)$ at one unit to the left of p. Then, by Proposition 2 and definition of g,

$$L([w_1, w_1 g(x)|x + 1]) = L([1, g(x)|x + 1]) = x,$$

i.e., L selects p with respect to $[w_1, w_1 g(x)|x + 1]$. Similarly, L selects p with respect to $[w_2 g(D - x), w_2 |D - x + 1]$. Invoking CS we deduce that L selects p with respect to

$[w_1, w_1 g(x)|x + 1] \cup [w_2 g(D - x), w_2 |D - x + 1]$. We illustrate this in Figure 2 for the case when $x, D - x > 1$.

Now choose $w' > w_1$ so that $w'g(x) = w_2 g(D - x)$. Place a weight of $w'g(x)$ at a distance of one unit to the right of p, and $w_2 g(D - x)$ at a distance of one unit to the left of p. By Proposition 3 and definition of g, $L([w'g(x), w_2 g(D - x)|2]) = 1$, i.e., L selects p with respect to $[w'g(x), w_2 g(D - x) |2]$. By assumption L selects p with respect to $[w_1, w_2 |D]$. Invoking CS we conclude that L selects p with respect to $[w'g(x), w_2 g(D - x) |2] \cup [w_1, w_2 |D]$. This violates PM, because $w' > w_1$ means that L should locate to the right of p. Hence, $L([w_1, w_2 |D]) = x \Rightarrow w_1 g(x) = w_2 g(D - x)$.

Now suppose there exists a $y \neq x$ such that $w_1 g(y) = w_2 g(D - y)$. Without loss of generality we may assume that $y < x$. From PM and CN we know that $h(w) = L([w_1, w |D])$ is a continuous increasing function of w. Now, $h(0) = 0$ by U (if $w = 0$, all the weight is at LEFT) and $h(w_2) = x$, by the definition of L. Hence, by the intermediate value theorem there must be a $w' \in [0, w_2]$ such that $h(w') = y$. Thus, $L([w_1, w' |D]) = y$. Hence, $w_1 g(y) = w'g(D - y)$. By the assumption about y this means that $w' = w_2$. Hence, $L([w_1, w_2 |D]) = y$, i.e., L is not single valued, a contradiction. □

The next proposition is needed to show that g is integrable.

Proposition 5. *The function g is strictly increasing.*

Proof. Suppose not. Then $\exists\, s_1 < s_2$ such that $g(s_1) \geq g(s_2)$. Suppose first that $g(s_1) = g(s_2) = w$, say. Then, by Proposition 3 and definition of g

$$L([w, w|s_1 + s_2]) = (s_1 + s_2)/2.$$

As $g(s_1) = g(s_2)$ it follows that $wg(s_1) = wg(s_1 + s_2 - s_1)$, which implies by Proposition 4 that L locates at distance s_1 from LEFT. As L is single valued, we conclude that $s_1 = (s_1 + s_2)/2$, i.e., $s_1 = s_2$, a contradiction.

Suppose now that $g(s_1) > g(s_2)$. We can assume that $g(s_2) \neq 0$, because $g(s_2) = 0$ for some $s_2 > 0$ would violate U (because all the weight would be at the point at distance $1 + s_2$ from LEFT). Consider the simple customer set $[w, 1|s_1 + s_2]$ where $w = g(s_1)/g(s_2) > 1$. Now, $wg(s_2) = g(s_1)$. So, by Proposition 4, $L([w, 1|s_1 + s_2]) = s_2 > (s_1 + s_2)/2 = L([1, 1|s_1 + s_2])$, which violates PM. This proves the result. □

Proposition 6. *For all simple customer sets $[w_1, w_2 |D]$ $\exists$ an nondecreasing, nonnegative, differentiable, strictly convex function $f(x) = \int_0^x g(t)dt$ such that:*

$$L([w_1, w_2 |D]) = \arg\min \{w_1 f(x) + w_2 f(D - x): x \in [0, D]\}.$$

Proof. As g is strictly increasing we know that $f(x) = \int_0^x g(t)dt$ exists and is nonnegative, nondecreasing, differentiable, and strictly convex (see, for example, Theorems 36 and 45 of Clapham 1973). The x that minimizes $w_1 f(x) + w_2 f(D - x)$ is unique and must satisfy (here f' is the derivative of f):

$$w_1 f'(x) - w_2 f'(D - x) = 0$$
$$\Rightarrow w_1 g(x) = w_2 g(D - x),$$
$$\Rightarrow \text{by Proposition 4 that } L([w_1, w_2 |D]) = x. \quad \square$$

Proposition 7. *Let L be any location rule that satisfies TI, CS, PM and CN. Then, $\exists$ a nonnegative, differentiable, nondecreasing, strictly convex function f such that:*

$$L([S, w(S)]) = \arg\min [\textstyle\sum \{w(y{:}S) f(d(x, y)): y \in S\} | x \in T],$$

for all customer sets $[S, w(S)]$.

Proof. Fix an L that satisfies TI, CS, PM, and CN. From Proposition 6 we know that $\exists$ an appropriate function f such that:

$$L([w_1, w_2 |D]) = \arg\min [w_1 f(x) + w_2 f(D - x) | x \in [0, D]].$$

Fix a tree T and customer set $[S, w(S)]$. Let $z = \arg\min [\Sigma\{w(y{:}S) f(d(x, y)): y \in S\} | x \in T]$. For each component C of $T \setminus z$ let

$$r(C) = \textstyle\sum \{w(y{:}S) f'(d(y, z)): y \in C \cap S\}.$$

What we will do is partition the components of $T \setminus z$ into two sets A and B, with possibly one component being 'split' between them. The partition is obtained from the solution of the following linear program:

$$\min \lambda,$$
$$\lambda \geq \textstyle\sum \{r(C)u(C): C \text{ a component of } T \setminus z\},$$
$$\lambda \geq \textstyle\sum \{r(C)[1 - u(C)]: C \text{ a component of } T \setminus z\},$$
$$0 \leq u(C) \leq 1 \text{ for each component } C \text{ of } T \setminus z.$$

Since the linear program has two constraints, there exists an optimal solution with at most one of the $u(C)$s being fractional. Let K be the component corresponding to this fractional variable. Furthermore, in an optimal solution

$$\lambda = (1/2) \textstyle\sum \{r(C): C \text{ a component of } T \setminus z\}.$$

Also,

$$\textstyle\sum \{r(C)u(C): C \text{ a component of } T \setminus z\}$$
$$= \textstyle\sum \{r(C)[1 - u(C)]: C \text{ a component of } T \setminus z\}.$$

We define two sets A and B as follows:

$$A = \{C: u(C) = 1\},$$
$$B = \{C: u(C) = 0\}.$$

Hence,

$$\textstyle\sum \{r(C): C \in A\} + u(K)r(K)$$
$$= \textstyle\sum \{r(C): C \in B\} + (1 - u(K))r(K).$$

Even though A and B are sets of components, we will treat them as sets of points. For example, a point x will be in A, if x is in a component C that is in A.

Also, the sets A and B will both be nonempty. Suppose not. Clearly, at least one of them must be nonempty. Let A be nonempty. Then all components except K must be in A. Hence:

$$\sum \{r(C)\colon C \in A\} + u(K)r(K) = (1 - u(K))r(K).$$

(If there were no fractional component, this equation would imply that $\Sigma\{r(C)\colon C \in A\} = 0$, a contradiction.) Rewriting this last equation, we deduce that

$$r(K) = \sum \{r(C)\colon C \in A\} + 2u(K)r(K)$$
$$> \sum \{r(C)\colon C \in A\},$$

which contradicts the fact that z is f'-stable with respect to S.

Let $[S, \lambda w(S)]$ be the customer set obtained by multiplying each weight in $[S, w(S)]$ by λ. By CS (see also Proposition 2)

$$L([S, \lambda w(S)]) = L([S, w(S)]),$$

and arg min $[\Sigma\{\lambda w(y{:}S)f(d(x, y))\colon y \in S\} \mid x \in T] =$ arg min $[\Sigma\{w(y{:}S)f(d(x, y))\colon y \in S\} \mid x \in T] = z$.

Let $S \backslash z = \{x_1, x_2, \ldots, x_k\}$ and $w(x_j{:}S) = w_j$ for $1 \leq j \leq k$. For each $x_p \in A$ and $x_q \in B$ define S_{pq} to be the customer set formed by assigning a weight of $w_p\, w_q f'(d(x_q, z))$ at x_p and a weight of $w_p\, w_q f'(d(x_p, z))$ at x_q. Notice that S_{pq} is well defined because A and B are nonempty. Observe that S_{pq} is a simple customer set and that:

$$[w_p w_q f'(d(x_q, z))]f'(d(x_p, z))$$
$$= [w_p w_q f'(d(x_p, z))]f'(d(x_q, z)).$$

Hence, by Proposition 4, $L(S_{pq}) = z$. Notice that this is true for any pair p and q such that $x_p \in A$ and $x_q \in B$. So, by CS, $L(\cup \{S_{pq}{:}x_p \in A, x_q \in B\}) = z$.

For each $x_p \in K$ and $x_q \in B$ define H_{pq} to be the customer set formed by assigning a weight of $u(K)w_p w_q f'(d(x_q, z))$ at x_p and a weight of $u(K)w_p w_q f'(d(x_p, z))$ at x_q. Notice that H_{pq} is a simple customer set and by Proposition 4, $L(H_{pq}) = z$.

For each $x_p \in K$ and $x_q \in A$ define G_{pq} to be the customer set formed by assigning a weight of $[1 - u(K)]w_p w_q f'(d(x_q, z))$ at x_p and a weight of $[1 - u(K)]w_p w_q f'(d(x_p, z))$ at x_q. Notice that G_{pq} is a simple customer set and by Proposition 4, $L(G_{pq}) = z$.

From consistency, it follows that the rule L assigns to the customer set

$$\Pi = \{G_{pq}\colon x_p \in K, x_q \in B\}$$
$$\cup \{H_{pq}{:}x_p \in K, x_q \in B\}$$
$$\cup \{S_{pq}{:}x_p \in A, x_q \in B\},$$

the location z.

We now show that the total weight assigned to a point x_p in the set Π is λw_p. Suppose first that x_p is in A. Then the weight that is assigned to it is

$$\sum \{w_p w_q f'(d(x_q, z)){:}x_q \in B\}$$
$$+ \sum \{[1 - u(K)]w_p w_q f'(d(x_q, z)){:}x_q \in K\}$$
$$= w_p \sum \{w_q f'(d(x_q, z)){:}x_q \in B\} + w_p[1 - u(K)]r(K)$$
$$= w_p[\sum \{r(C)\colon C \in B\} + [1 - u(K)]r(K)] = \lambda w_p.$$

A similar argument applies when x_p is in B and K. Thus, if $w(z{:}S) = 0$, $\Pi = [S, \lambda w(S)]$. Hence $L([S, \lambda w(S)]) = z$, i.e., $L([S, w(S)]) = z$. If $w(z{:}S) > 0$ then, by U, $L([z, \lambda w(z{:}S)]) = z$. So, by consistency

$$L(\Pi \cup [z, w(z{:}S)]) = z.$$

However,

$$[S, \lambda w(S)] = \Pi \cup [z, \lambda w(S, z)],$$

thus $L([S, w(S)]) = z$. □

To show where each axiom is explicitly used in the proof of the propositions, see Table I. Proposition 6 is not listed because it relies on the earlier propositions. Proposition 3 relies only on TI. Finally, all statements about simple customer sets make use of TI.

Table I

Proposition	Axioms Used
Proposition 2	CS and CN
Proposition 4	TI, CS, CN, and PM
Proposition 5	TI and PM
Proposition 7	CS

POSSIBLE EXTENSIONS

In this section we discuss some of the obstacles that must be surmounted in order to generalize the result to arbitrary graphs. In moving from trees to arbitrary graphs, one must relax the requirement that $L([S, w(S)])$ is single valued. To see why, take as our underlying graph an equilateral triangle with a single customer at each corner. The symmetry inherent in this structure ensures that for each point there is another that "looks" just like it. This eliminates a location rule that is single valued. Thus, the statement of the axioms needs to be modified. As we have mentioned earlier, this is possible. For example, in the continuity axiom we would impose Hausdorff continuity. In the case of trees, the argument presented here is easily extended to deal with this possibility, but there are many technicalities having to do with existence and uniqueness of functions and integrals. These technicalities multiply when one moves from trees to general graphs.

Even if these purely technical issues can be finessed, there is another hurdle to be cleared. That is, Proposition 1 does not hold for general graphs. This proposition can be interpreted as a kind of local optimality implies global optimality theorem and is related to the fact that the distance functions $d(x, y)$ are convex (Dearing et al. 1976 show that this convexity property holds if and only if the underlying network is a tree). The reader will recall that we made heavy use of this. In effect we showed that the axioms implied a location that satisfied the local optimality conditions and then invoked Proposition 1 to complete the proof. The theory of variational inequalities tells us that such theorems hold whenever the underlying space has a fixed point property. Certainly trees have such a property (this follows from a fixed point theorem of Eilenberg and Montgomery 1946), but general graphs do not (consider the function that rotates every point on a cycle one radian to the right). Absent a (nonaxiomatic) characterization of the appropriate class of locations, further research is needed.

ACKNOWLEDGMENTS

This research was supported in part by a Dean's Research Professorship from the College of Business of the Ohio State University. This paper has benefited greatly from the suggestions and advice of the referees. Thanks to Georgia Perakis for pointing out the connections to the theory of variational inequalities.

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